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Abdus Saboor
Kohat University of Science and Technology

Gauss Moutinho Cordeiro
Federal University of Pernambuco

Muhammad Nauman Khan
Kohat University of Science and Technology

Serge B. Provost
The University of Western Ontario

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The Transmuted Generalized Modified Weibull Distribution

Gauss M. Cordeiro[a], Abdus Saboor[b], Muhammad Nauman Khan[b], Serge B. Provost[c], Edwin M.M. Ortega[d]

[a] *Departamento de Estatística, Universidade Federal de Pernambuco, 50740-540, Recife, PE, Brazil*
[b] *Department of Mathematics, Kohat University of Science & Technology, Kohat, Pakistan 26000*
[c] *Department of Statistical & Actuarial Sciences, the University of Western Ontario, London, N6A5B7 Canada*
[d] *Departamento de Ciłncias Exatas, Universidade de São Paulo, 13418-900, Piracicaba, SP Brazil.*

Abstract. A profusion of new classes of distributions has recently proven useful to applied statisticians working in various areas of scientific investigation. Generalizing existing distributions by adding shape parameters leads to more flexible models. We define a new lifetime model called the transmuted generalized modified Weibull distribution from the family proposed by Aryal and Tsokos [1], which has a bathtub shaped hazard rate function. Some structural properties of the new model are investigated. The parameters of this distribution are estimated using the maximum likelihood approach. The proposed model turns out to be quite flexible for analyzing positive data. In fact, it can provide better fits than related distributions as measured by the Anderson-Darling (A^*) and Cramér-von Mises (W^*) statistics, which is illustrated by applying it to two real data sets. It may serve as a viable alternative to other distributions for modeling positive data arising in several fields of science such as hydrology, biostatistics, meteorology and engineering.

1. Introduction

The Weibull distribution is a popular lifetime model in reliability engineering. However, this distribution does not have a bathtub or upside–down bathtub shaped hazard rate function (hrf), and thus cannot be used to model the lifetime of certain systems. To overcome this shortcoming, several generalizations of the classical Weibull distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining better fits. Extensions of the Weibull distribution have been developed for one or more of the following reasons: providing a physical or statistical theoretical interpretation to explain the mechanism whereby the data is generated, improving on a model that has previously been used successfully, and introducing a model whose empirical fit better suits a particular data set.

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Email addresses: gausscordeiro@gmail.com (Gauss M. Cordeiro), saboorhangu@gmail.com; dr.abdussaboor@kust.edu.pk (Abdus Saboor), zaybasdf@gmail.com (Muhammad Nauman Khan), provost@stats.uwo.ca (Serge B. Provost), edwin@usp.br (Edwin M.M. Ortega)

The generalized Weibull distributions arise in numerous areas of research as can been seen for instance from the papers of Aryal and Tsokos [1], Lai *et al.* [2], Ghitany *et al.* [3], Carrasco *et al.* [4], Saboor *et al.* [5], Cordeiro *et al.* [6], Cordeiro *et al.* [7], Peng and Yan [8], Khan [9], Tojeiro *et al.* [10], Almalki and Yuan [11] and Saboor *et al.* [12]. Many extended Weibull models have upside–down bathtub shaped hazard rates, which is the case for the extensions discussed by Carrasco *et al.* [4], Jiang and Murthy [13] and Silva *et al.* [14], among others. Certainly, one of the most important extensions is the modified Weibull (MW) distribution introduced by Lai *et al.* [2]. Let $G_L(x;\alpha,\gamma,\lambda)$ be the MW cumulative distribution function (cdf) with the three parameters $\alpha > 0$, $\gamma > 0$ and $\lambda \geq 0$. Carrasco *et al.* [4] proposed the generalized modified Weibull (GMW) distribution by raising $G(x;\alpha,\gamma,\lambda)$ to the power $\varphi > 0$, say $G(x) = G_L(x;\alpha,\gamma,\lambda)^{\varphi}$. Thus, its cdf is given by

$$G(x) = [1 - \exp\{-\alpha x^{\gamma} \exp(\lambda x)\}]^{\varphi}. \tag{1}$$

The cdf (1) is sometimes referred to as that of the *exponentiated modified Weibull* distribution. Its associated probability density function (pdf) is

$$g(x) = \frac{\alpha\varphi x^{\gamma-1}\,(\gamma + \lambda x)\,\exp\{\lambda x - \alpha x^{\gamma}\exp(\lambda x)\}}{[1 - \exp\{-\alpha x^{\gamma}\exp(\lambda x)\}]^{1-\varphi}},\ \ x > 0. \tag{2}$$

The parameter α is a scaling parameter, whereas the parameters γ and φ affect the shape of the distribution. The parameter λ is a sort of accelerating factor with respect to time which is acting increasingly as a fragility factor in the survival of a unit. The MW distribution corresponds to $\varphi = 1$. The Weibull distribution is a special model of (2) with $\lambda = 0$ and $\varphi = 1$. If Z is a random variable with density function (2), we write $Z \sim \mathrm{GMW}(\alpha,\gamma,\lambda,\varphi)$.

Adding new shape parameters to expand a model into a larger family of distributions that can be significantly skewed and heavy-tailed plays a fundamental role in distribution theory. More recently, new univariate continuous distributions have been defined by introducing additional shape parameters to a baseline model. Indeed, there has been an increased interest in defining new generators for univariate continuous distributions by making use of this technique. The introduction of additional parameter(s) has proved useful in exploring tail properties and for improving the goodness-of-fit of the proposed generator family as well. Aryal and Tsokos [1] pursued this approach by proposing an interesting method for adding a new parameter to an existing distribution. The resulting distribution provides more flexibility to model various types of data. It can be defined as follows: If the baseline distribution has the cdf $G(x)$ and pdf $g(x)$, the cdf and pdf (for $|\eta| \leq 1$) of the *transmuted extended distribution* are given by

$$F(x) = (\eta + 1)\,G(x) - \eta\,G(x)^2 \tag{3}$$

and

$$f(x) = (1 + \eta)\,g(x) - 2\,\eta\,G(x)\,g(x), \tag{4}$$

respectively.

Note that $\eta = 0$ in equation (4) corresponds to the baseline distribution. Further details can be found in Shaw and Buckley [15].

Some distributions belonging to this class have recently been introduced. For example, Aryal and Tsokos [1] defined the transmuted Weibull for modeling the tensile fatigue characteristics of a polyester/viscose yarn. We now generalize that model by applying the transmuted technique [1] to equation (2), which defines the so-called *transmuted generalized modified Weibull* (TGMW) distribution. Then, the cdf, survival function, pdf and hrf of this distribution are obtained, respectively, from equations (3) and (4) (for $x > 0$, $\alpha > 0$, $\gamma > 0$, $\lambda > 0$, $\varphi > 0$, $|\eta| \leq 1$) as

$$F(x) = [1 - \exp\{-\exp(x\lambda)x^{\gamma}\alpha\}]^{\varphi}\left(1 + \eta - \eta\,[1 - \exp\{-\exp(x\lambda)x^{\gamma}\alpha\}]^{\varphi}\right), \tag{5}$$

$$S(x) = 1 - [1 - \exp\{-\exp(x\lambda)x^{\gamma}\alpha\}]^{\varphi}\left(1 + \eta - \eta\,[1 - \exp\{-\exp(x\lambda)x^{\gamma}\alpha\}]^{\varphi}\right), \tag{6}$$

$$f(x) = \alpha\varphi x^{\gamma-1}(x\lambda+\gamma)\exp\{x\lambda - \alpha x^{\gamma}\exp(x\lambda)\}\left[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}\right]^{\varphi-1} \times \Big(1+\eta-2\eta\left[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}\right]^{\varphi}\Big) \tag{7}$$

and

$$h(x) = \alpha\varphi x^{\gamma-1}(x\lambda+\gamma)\exp\{x\lambda - \alpha x^{\gamma}\exp(x\lambda)\}\left[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}\right]^{\varphi-1} \times \frac{\Big(1+\eta-2\eta\left[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}\right]^{\varphi}\Big)}{1-\left[1-\exp\{-\exp(x\lambda)x^{\gamma}\alpha\}\right]^{\phi}\Big(1+\eta-\eta\left[1-\exp\{-\exp(x\lambda)x^{\gamma}\alpha\}\right]^{\varphi}\Big)}. \tag{8}$$

Henceforth, a random variable X having density function (7) is denoted by X ∼TGMW$(\alpha,\varphi,\gamma,\lambda,\eta)$. We note that the parameters of the GMW model in equation (1) are identifiable, i.e., if $\theta_1 = (\alpha_1,\gamma_1,\lambda_1,\varphi_1)^T$ and $\theta_2 = (\alpha_2,\gamma_2,\lambda_2,\varphi_2)^T$ are such that if $\theta_1 \neq \theta_2$ then $G_{\theta_1}(x) \neq G_{\theta_2}(x)$ for all $x > 0$. Due to the identifiability of the GMW distribution, the transmuted extended generator (3) guarantees the identifiability of the proposed distribution. A further motivation for the TGMW model is to test its adequacy versus several sub-models on a data set using the classical likelihood ratio statistics.

The paper is organized as follows. Some special models of the new distribution are provided in Section 2. Plots of the parameter effects on the pdf (7) are given in Section 3. Useful expansions for (5) and (7) are derived in Section 4. Explicit expressions for certain statistical functions of X are obtained in Section 5. The estimation of the model parameters is addressed in Section 6. In Section 7, we illustrate the usefulness of the TGMW distribution for modeling lifetime data by applying it to two real data sets originating from the engineering and biological sciences, and comparing this new model to some other related distributions. Finally, Section 8 offers some concluding remarks.

2. Special Distributions

The TGMW distribution includes as sub-models the following known distributions:

- **Generalized Modified Weibull distribution**

  For $\eta = 0$, equation (7) yields (2).

- **Weibull distribution**

  For $\lambda = 0$, $\varphi = 1$ and $\eta = 0$, equation (7) yields

  $$f(x) = \alpha\gamma x^{\gamma-1}\exp(-\alpha x^{\gamma}),\ x > 0,$$

  which is the classical two-parameter Weibull density. If $\gamma = 1$ and $\gamma = 2$ in addition to $\lambda = 0$ and $\varphi = 1$, it gives the exponential and Rayleigh distributions, respectively.

- **Extreme-value distribution**

  For $\gamma = 0$, $\varphi = 1$ and $\eta = 0$, equation (7) gives

  $$f(x) = \alpha\lambda\exp\Big\{\lambda x - \alpha\exp(\lambda x)\Big\},\ x > 0,$$

  which is a type I extreme-value (also known as the log-gamma) density.

- **Exponentiated Weibull distribution**

  For $\lambda = 0$ and $\eta = 0$, the TGMW density reduces to

  $$f(x) = \alpha\varphi\gamma x^{\gamma-1}\exp(-\alpha x^{\gamma})\Big\{1-\exp(-\alpha x^{\gamma})\Big\}^{\varphi-1},\ x > 0,$$

which is the EW density introduced by Mudholkar *et al.* [16] and Mudholkar *et al.* [17]. If $\gamma = 1$ in addition to $\lambda = 0$ and $\eta = 0$, the TGMW distribution becomes the exponentiated exponential due to Gupta and Kundu [18] and Gupta and Kundu [19]. If $\gamma = 2$ in addition to $\lambda = 0$ and $\eta = 0$, it reduces to the generalized Rayleigh distribution [20].

- **Modified Weibull distribution**

  For $\varphi = 1$ and $\eta = 0$, the TGMW density becomes

$$f(x) = \alpha x^{\gamma-1}(\gamma + \lambda x)\exp\left\{\lambda x - \alpha x^{\gamma}\exp(\lambda x)\right\},\ x > 0,$$

  which is the MW density introduced by Lai *et al.* [2].

- **Beta integrated distribution**

  The beta integrated distribution was defined by Lai *et al.* [21] and its survival function is

$$S(x) = \exp\left\{-\alpha x^{\gamma}(1 - dx)^{c}\right\}, \quad \alpha, \gamma, d > 0, c < 0.$$

  Setting $d = 1/n$ and $c = -n\lambda$ and letting $n \to \infty$, one has

$$(1 - dx)^{c} = \left(1 - \frac{x}{n}\right)^{-\lambda n} \to \exp(\lambda x)$$

  and, in the limit,

$$S(x) \approx \exp\left\{-\alpha x^{\gamma}\exp(\lambda x)\right\},$$

  which is the survival function (6) of the TGMW distribution when $\varphi = 1$ and $\eta = 0$.

3. The Effect of the Parameters on the pdf and hrf

Plots of the pdf (7) and hrf (8) of the TGMW distribution are displayed in Figures 1-4 for selected parameter values.

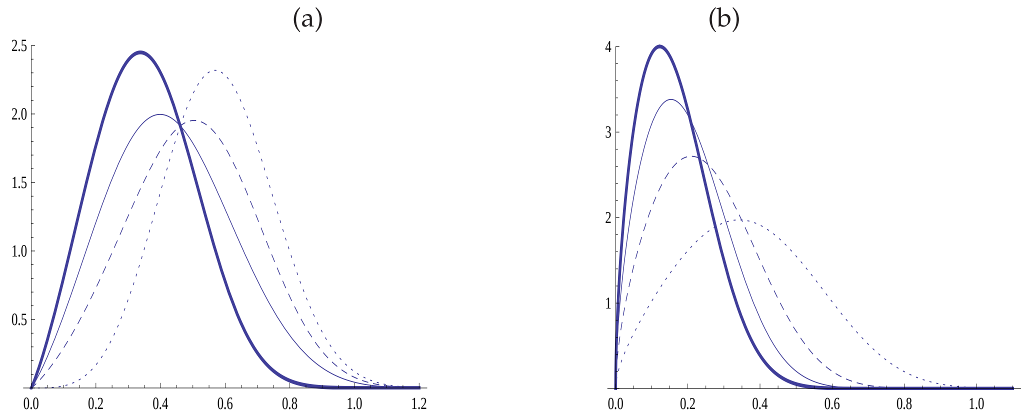

Figure 1: Plots of the TGMW density function. (a) $\alpha = 1.5$; $\varphi = 1.6$; $\gamma = 1.3$; $\lambda = 1.4$ and $\eta = -1$ (dotted line), $\eta = -0.3$ (dashed line), $\eta = 0.3$ (solid line), $\eta = 1$ (thick line). (b) $\varphi = 1.2$; $\gamma = 1.3$; $\lambda = 1.4$; $\eta = 1$ and $\alpha = 1$ (dotted line), $\alpha = 2$ (dashed line), $\alpha = 3$ (solid line), $\alpha = 4$ (thick line).

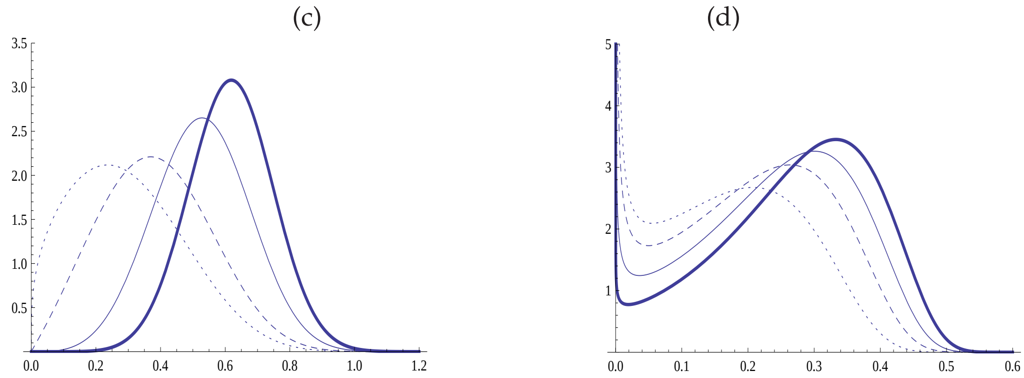

Figure 2: Plots of the TGMW density function. (c) $\alpha = 1.2$; $\gamma = 1.3$; $\lambda = 1.4$; $\eta = 1$ and $\varphi = 1$ (dotted line), $\varphi = 1.5$ (dashed line), $\varphi = 3$ (solid line), $\varphi = 5$ (thick line). (d) $\alpha = 0.1$; $\varphi = 0.5$; $\lambda = 10$; $\eta = 0.13$ and $\gamma = 0.3$ (dotted line), $\gamma = 0.8$ (dashed line), $\gamma = 1.2$ (solid line), $\gamma = 1.6$ (thick line).

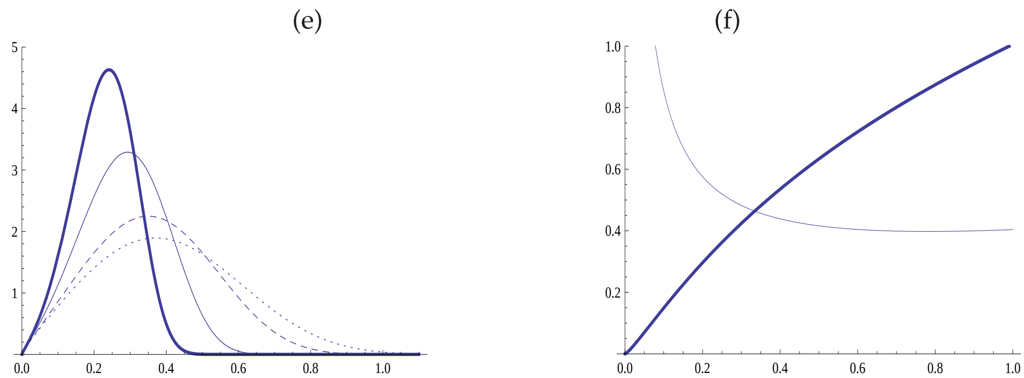

Figure 3: Plots of the TGMW density function. (e) $\alpha = 1.2$; $\gamma = 1.4$; $\varphi = 1.3$; $\eta = 1$ and $\lambda = 1$ (dotted line), $\lambda = 1.5$ (dashed line), $\lambda = 3$ (solid line), $\lambda = 5$ (thick line). (f) The TGMW hazard rate function. $\alpha = 1.2$; $\varphi = 3.3$; $\gamma = 0.7$; $\lambda = 0.2$; $\eta = 0.1$ (increasing hrf) and $\alpha = 0.5$; $\varphi = 1.5$; $\gamma = 0.14$; $\lambda = 0.25$; $\eta = 1$ (decreasing hrf).

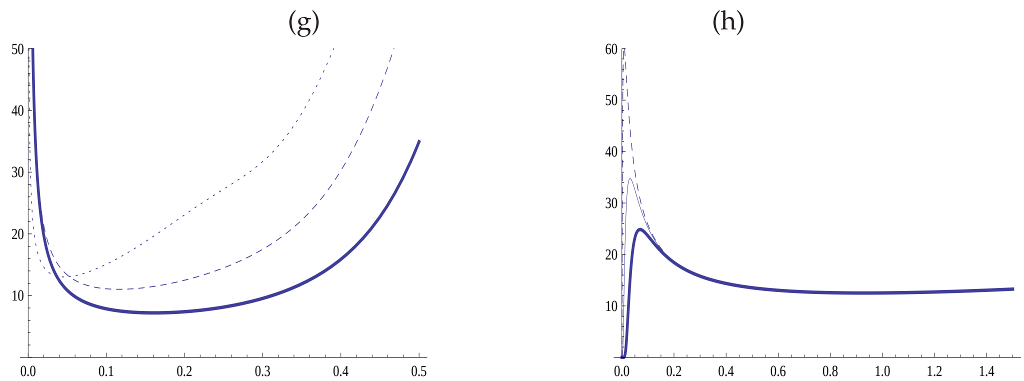

Figure 4: Plots of the TGMW hrf. (g) $\alpha = 1.8$; $\lambda = 5.2$; $\eta = 0.9$ and $\gamma = 0.7$; $\varphi = 0.7$ (dotted line), $\gamma = 0.07$; $\varphi = 1.7$ (dashed line), $\gamma = 0.01$; $\varphi = 3.0$ (thick line). (f) The TGMW hazard rate function. $\alpha = 15$; $\lambda = 0.25$; $\eta = 0.01$; $\gamma = 0.4$ and $\varphi = 10$ (dotted line), $\varphi = 30$ (dashed line), $\varphi = 90$ (thick line).

The plots in Figures 1, 2 and 3 (e) reveal how the parameters α, φ, γ, λ and η affect the TGMW density. They illustrate the flexibility of the new distribution. The plots in Figures 3 (f) and 4 indicate that the hrf of the TGMW distribution can take the most common forms in real applications: increasing, decreasing, bathtub and upside-down bathtub shapes.

4. Useful Expansions

In this section, we demonstrate that the TGMW density (7) can be expressed as a linear mixture of MW densities.

The cdf (5) of the TGMW distribution can be written as

$$F(x) = (1+\eta)\,[1-\exp\{-\exp(x\lambda)\,x^{\gamma}\,\alpha\}]^{\phi} - \eta\,[1-\exp\{-\exp(x\lambda)x^{\gamma}\alpha\}]^{2\phi}\,. \tag{9}$$

It is then seen that this cdf is simply a linear combination of two GMW cdf's. Similarly, the corresponding pdf is a linear combination of two GMW pdf's.

In light of the generalized binomial expansion

$$(1-x)^{\alpha} = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}x^k\,,$$

equation (9) can be further rewritten as

$$F(x) = \sum_{k=0}^{\infty} e_k\,\exp\{-k\,\alpha\exp(x\lambda)x^{\gamma}\}, \tag{10}$$

where

$$e_k = (-1)^k\left[(1+\eta)\binom{\phi}{k} - \eta\binom{2\phi}{n}\right].$$

Let $S_{k+1}(x) = \exp\{-[k+1]\,\alpha\exp(x\lambda)x^{\gamma}\}$ (for $k \geq 0$) be the survival function of the MW$([k+1]\alpha, \gamma, \lambda)$ distribution and $h_{k+1}(x)$ be the corresponding density function. By differentiating (10), we obtain

$$f(x) = \sum_{k=0}^{\infty} d_{k+1}\,h_{k+1}(x), \tag{11}$$

where $d_{k+1} = -e_{k+1}$ for $k \geq 0$, which constitutes the main result of this section. It can be used to obtain some mathematical properties of the TGMW distribution from those of the MW properties.

5. Statistical Functions

5.1. Moments

In this section, we explain how to evaluate the positive, negative, central and factorial moments of a TGMW random variable denoted by X. The rth ordinary real moment of X, say $\mu'_r = E(X^r)$, follows from (7) as

$$\mu'_r = \alpha\,\varphi \int_0^{\infty} x^r\, x^{\gamma-1}(x\lambda+\gamma)\exp\{x\lambda-\alpha x^{\gamma}\exp(x\lambda)\}\,[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}]^{\varphi-1}$$
$$\times\Big(1+\eta-2\eta\,[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}]^{\varphi}\Big)\,\mathrm{d}x.$$

This expression can be rewritten as

$$\mu'_r = \alpha\,\varphi\,(1+\eta)\int_0^{\infty} x^{r+\gamma-1}\,(x\lambda+\gamma)\exp\{x\lambda-\alpha x^{\gamma}\exp(x\lambda)\}\times[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}]^{\varphi-1}\,\mathrm{d}x$$
$$-\,2\,\alpha\,\varphi\,\eta\int_0^{\infty} x^{r+\gamma-1}\,(x\lambda+\gamma)\exp\{x\lambda-\alpha x^{\gamma}\exp(x\lambda)\}\times[1-\exp\{-\alpha x^{\gamma}\exp(x\lambda)\}]^{2\varphi-1}\,\mathrm{d}x\,. \tag{12}$$

We recommend making use of numerical integration to evaluate these moments. In our experience, such calculations are quick and accurate. Moreover, they should be less prone to computational errors than complicated analytical truncated closed-form representations. Table 1 provides some numerical values for the first four moments, as well as the skewness and kurtosis of X based on equation (12) for selected parameter values.

Table 1: Moments, Skewness and Kurtosis of X for selected parameter values

Moments →	μ'_1	μ'_2	μ'_3	μ'_4	Skewness	Kurtosis
$\eta\downarrow$	$\alpha=0.2$	$\varphi=1.3$	$\gamma=0.6$	$\lambda=0.5$		
-0.8	2.59897	7.76566	25.2754	87.6567	-0.15981	2.68669
-0.2	2.21679	6.14381	19.1001	64.2293	0.021102	2.36019
0.2	1.962	5.06258	14.9832	48.6111	0.217101	2.37599
0.5	1.77091	4.25166	11.8956	36.8974	0.352412	2.51519
0.8	1.57982	3.44073	8.80791	25.1837	0.420885	2.64489
$\lambda\downarrow$	$\alpha=0.5$	$\varphi=0.3$	$\gamma=1.6$	$\eta=0.4$		
0.2	0.441272	0.476839	0.716051	1.29834	1.71278	6.0032
0.8	0.344198	0.252273	0.240564	0.266139	1.25911	4.03014
1.4	0.293484	0.171431	0.126824	0.107217	1.0615	3.39105
1.8	0.269978	0.140661	0.091753	0.067963	0.973929	3.14709
3.0	0.222773	0.090003	0.044537	0.024730	0.800812	2.7331
$\gamma\downarrow$	$\alpha=2.5$	$\varphi=0.7$	$\lambda=0.1$	$\eta=-0.4$		
0.2	0.202613	0.478456	1.83756	9.06083	5.40426	40.1645
0.7	0.292455	0.23941	0.313669	0.552059	2.54536	12.0798
1.5	0.451737	0.302441	0.253996	0.250493	0.923425	3.81448
3.0	0.623854	0.448197	0.353979	0.300299	0.052411	2.6342
5.0	0.736295	0.576583	0.472684	0.401894	-0.404401	2.97977
$\varphi\downarrow$	$\alpha=0.9$	$\gamma=0.4$	$\lambda=2.1$	$\eta=-0.7$		
0.1	0.056051	0.021173	0.010601	0.006174	3.05372	12.8168
0.5	0.211323	0.090457	0.047919	0.028817	0.963809	3.13018
1.1	0.340613	0.164441	0.092710	0.057909	0.348364	2.42396
1.7	0.417044	0.217833	0.128553	0.082714	0.117802	2.50887
2.5	0.482564	0.270831	0.167252	0.111013	-0.01088	2.70552
$\alpha\downarrow$	$\varphi=3.3$	$\gamma=2.0$	$\lambda=0.9$	$\eta=0.5$		
0.1	1.71328	3.02534	5.49102	10.2218	-0.02385	3.06834
0.7	0.929461	0.904557	0.91756	0.966692	0.149935	3.05741
2.5	0.57937	0.355553	0.229803	0.155734	0.273581	3.12829
5.0	0.438001	0.204438	0.101053	0.052642	0.337649	3.19065
7.0	0.380448	0.154668	0.066759	0.030441	0.366921	3.22503

The hth negative real moment can be determined by replacing r with $-h$ in equation (12).

Further, the central moments (μ_n) and cumulants (κ_n) of X are easily determined from (12) as

$$\mu_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu_1'^{k}\,\mu'_{n-k}\qquad\text{and}\qquad \kappa_n=\mu'_n-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\kappa_k\,\mu'_{n-k},$$

respectively, where $\kappa_1=\mu'_1$. Thus, $\kappa_2=\mu'_2-\mu_1'^{2}$, $\kappa_3=\mu'_3-3\mu'_2\mu'_1+2\mu_1'^{3}$, etc. Clearly, the skewness and kurtosis of X can be obtained from the ordinary moments using well-known formulae.

Finally, the nth descending factorial moment of X (for $n=1,2,\ldots$) is

$$\mu'_{(n)}=E[X^{(n)}]=E[X(X-1)\times\cdots\times(X-n+1)]=\sum_{j=0}^{n}s(n,j)\,\mu'_j,$$

where $s(n, j) = (j!)^{-1}\,[d^j j^{(n)}/dx^j]_{x=0}$ is the Stirling number of first kind.

5.2. Reliability

In the area of stress-strength models, there has been a large amount of work as regards the estimation of the reliability $R = Pr(X_2 < X_1)$ when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions. The algebraic form for R has been worked out for the majority of the well-known standard distributions. We now derive the reliability R when X_1 and X_2 are independent TGMW random variables with the same parameters except for the parameters η_1 and η_2, respectively.

Theorem 5.1. *The reliability $R = Pr(X_2 < X_1)$ is the solution of the equation*

$$R = \int_0^\infty f_1(x;\alpha,\gamma,\lambda,\varphi,\eta_1)\,F_2(x;\alpha,\gamma,\lambda,\varphi,\eta_2)\mathrm{d}x\,, \tag{13}$$

which is given in equation (15).

Proof. Substituting (5) and (7) in equation (13), we obtain

$$\begin{aligned} R = \alpha\varphi \int_0^\infty & x^{\gamma-1}(x\lambda+\gamma)\exp\{x\lambda - \alpha x^\gamma \exp(x\lambda)\}\,[1-\exp\{-\alpha x^\gamma \exp(x\lambda)\}]^{\varphi-1} \\ & \times\Big(1+\eta_1 - 2\eta_1\,[1-\exp\{-\alpha x^\gamma \exp(x\lambda)\}]^\varphi\Big)\,[1-\exp\{-\exp(x\lambda)x^\gamma\alpha\}]^\varphi \\ & \times\Big(1+\eta_2 - [1-\exp\{-\exp(x\lambda)x^\gamma\alpha\}]^\varphi\,\eta_2\Big)\mathrm{d}x. \end{aligned}$$

Letting $u = \exp(-\alpha x^\gamma \exp(\lambda x))$, $\frac{1}{u}\,\mathrm{du} = -\alpha \mathrm{x}^\gamma \exp(\lambda \mathrm{x})[(\gamma+\lambda \mathrm{x})/\mathrm{x}]\mathrm{dx}$ and we have

$$R = -\varphi \int_0^1 (1-u)^{2\varphi-1}\{1+\eta_1 - 2\eta_1(1-u)^\varphi\}\{1+\eta_2-\eta_2(1-u)^\varphi\}\mathrm{d}u\,. \tag{14}$$

Solving the integral in (14), we obtain

$$R = \frac{\eta_1-\eta_2-3}{6}\,. \tag{15}$$

□

5.3. Order statistics

The density of the ith order statistic $X_{i:n}$, $f_{i:n}(x)$ say, in a random sample of size n from the TGMW distribution is given by

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}\,f(x)\,F(x)^{i-1}\,[1-F(x)]^{n-i}, \quad i=1,\ldots,n, \tag{16}$$

where $B(\cdot,\cdot)$ is the beta function.

Theorem 5.2. *The density of $X_{i:n}$ can be expressed as the mixture of MW densities specified by equation (19).*

Proof. The derivation is based on an equation of Section 0.314 of Gradshteyn and Ryzhik [23] for a power series raised to a positive integer j

$$\left(\sum_{s=0}^\infty a_s\,u^s\right)^j = \sum_{s=0}^\infty c_{j,s}\,u^s, \tag{17}$$

where $c_{j,0} = a_0^j$ and the coefficients $c_{j,s}$ (for $s \geq 1$) are obtained from the recurrence equation

$$c_{j,s} = (s\,a_0)^{-1} \sum_{m=1}^{s} [m(j+1) - s]\, a_m\, c_{j,s-m}. \tag{18}$$

The coefficients $c_{j,s}$ can be evaluated numerically from (18) and the constants $a_0, \ldots, a_s$, by using any computing software packages such as *Matlab*, *Maple* or *Mathematica*.

We can rewrite (16) as

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)}\, f(x) \sum_{k=0}^{i-1} (-1)^k\, [1 - F(x)]^{n-i+k}.$$

We define $a_s = d_{s+1}$ (for $s \geq 0$) and $u = u(x) = \exp\{-\alpha\, x^{\gamma}\, \exp(\lambda x)\}$ as in Section 5.2. Since $S(x) = \sum_{s=0}^{\infty} a_s\, u^{s+1}$, the above sum becomes

$$\sum_{k=0}^{i-1} (-1)^k \Big(\sum_{s=0}^{\infty} a_s\, u^{s+1} \Big)^{n-i+k} = \sum_{k=0}^{i-1} (-1)^k \sum_{s=0}^{\infty} c_{n-i+k,s}\, u^{(n-i+k)+s},$$

where the constants $c_{n-i+k,s}$ can be determined from the a_s's using (18).

Based on the mixture form (11), we can write

$$\begin{aligned} f_{i:n}(x) \;&=\; \frac{1}{B(i, n-i+1)} \sum_{j,s=0}^{\infty} \sum_{k=0}^{i-1} (-1)^k\, a_j\, c_{n-i+k,s}\, (j+1)\, \alpha\, x^{\gamma-1}\, (\gamma + \lambda x) \\ &\quad \times \exp\{\lambda x - \alpha\, [n-i+k+1+s+j]\, x^{\gamma}\, \exp(\lambda x)\}\,. \end{aligned}$$

Finally, we obtain

$$f_{i:n}(x) \;=\; \sum_{k=0}^{i-1} \sum_{j,s=0}^{\infty} p_{k,j,s}\, h_{n-i+k+1+s+j}(x), \tag{19}$$

where $h_{n-i+k+1+s+j}(x)$ is the MW($[n-i+k+1+s+j]\alpha, \gamma, \lambda$) density function and

$$p_{k,j,s} = p_{k,j,s}(a, b, i, n) = \frac{(-1)^k\, (j+1)\, a_j\, c_{n-i+k,s}}{(n-i+k+1+s+j)}.$$

□

Some mathematical properties of the TGMW order statistics can be obtained from (19) and the properties of the MW distribution.

5.4. Quantile Function

The quantile function (qf) is useful for determining various mathematical properties of a distribution. In some cases, it is possible to invert the cdf. However, for some other distributions, the inverse function cannot be obtained in closed-form. In the case at hand, we shall resort to power series methods, which are at the heart of many solutions in applied mathematics and statistics. The qf $Q_X(p)$ of the random variable $X \sim F$ is defined from the generalized inverse of its cdf for a fixed probability p, namely

$$Q_X(p) = \inf\{x \in \mathbb{R}^+ : p \leq F(x)\}, \qquad p \in (0,1).$$

For the TGMW distribution, one has the following result:

Theorem 5.3. *Let $X \sim TGMW(\alpha, \varphi, \gamma, \lambda, \eta)$. Then the related quantile function $Q_X(p)$ is the unique positive solution of the equation*

$$1 - \exp\{-\alpha\, x^{\gamma}\, \exp(\lambda x)\} = (A_{1,2})^{\frac{1}{\phi}}\,,$$

which is given in equation (21).

Proof. We have to invert the equation $F(x) = p$ for some fixed $p \in (0, 1)$ with respect to x. Setting

$$A = [1 - \exp\{-\alpha\, x^{\gamma}\, \exp(\lambda x)\}]^{\phi}\,,$$

the problem reduces to solving the quadratic equation $\eta A^2 - (1 + \eta)A + p = 0$. Thus,

$$A_{1,2} = \frac{1 + \eta \pm \sqrt{(1+\eta)^2 - 4p\eta}}{2\eta}\,.$$

We look for an explicit solution $x = Q_X(p)$ of the nonlinear equation

$$1 - \exp\{-\alpha\, x^{\gamma}\, \exp(\lambda x)\} = (A_{1,2})^{\frac{1}{\phi}}.$$

Since the left-hand side of this equation is less then one, both solutions $A_{1,2}$ cannot be satisfactory for this model. Actually, we have the restriction $A < 1$ on the whole range of parameters $\min(\alpha, \gamma, \lambda, \phi) > 0$ in conjunction with $p \in (0, 1)$. Since

$$A_1 - 1 = \frac{1 - \eta + \sqrt{(1+\eta)^2 - 4p\eta}}{2\eta} < \frac{1 - \eta + 1 + \eta}{2\eta} = \frac{1}{\eta}\,, \quad \eta > 0\,,$$

and

$$A_2 - 1 = \frac{1 - \eta - \sqrt{(1+\eta)^2 - 4p\eta}}{2\eta} < \frac{1 - \eta - (1 - \eta)}{2\eta} = 0\,, \quad \eta > 0\,,$$

we obtain

$$1 - \exp\{-\alpha\, x^{\gamma}\, \exp(\lambda x)\} = A^{\frac{1}{\phi}} = \left(\frac{1 - \eta - \sqrt{(1+\eta)^2 - 4p\eta}}{2\eta}\right)^{\frac{1}{\phi}}\,, \quad \eta > 0\,, \tag{20}$$

to be solved in x. Further, we need the Lambert W-function, the inverse function of $W \mapsto W\exp(W)$. Its principal branch W_P is the solution of $W\exp(W) = x$ for which $W_P(x) \geq W_P\{-\exp(-1)\}$. This function is implemented in *Mathematica* as `ProductLog[z]`. We are interested in W_P exclusively for $x > 0$, where it is single–valued and monotone increasing, see [22].

Next, it follows that

$$\exp\{-\alpha\, x^{\gamma}\, \exp(\lambda x)\} = 1 - A^{\frac{1}{\phi}}, \quad x^{\gamma}\, \exp(\lambda x) = \log\left(1 - A^{\frac{1}{\phi}}\right)^{-\frac{1}{\alpha}},$$

$$(\lambda x)^{\gamma}\, \exp(\lambda x) = \frac{\lambda^{\gamma}}{\alpha}\, \log\frac{1}{1 - A^{\frac{1}{\phi}}}, \quad \frac{\lambda x}{\gamma}\, \exp\left(\frac{\lambda x}{\gamma}\right) = \frac{\lambda}{\gamma \alpha^{\frac{1}{\gamma}}}\left[\log\left(\frac{1}{1 - A^{\frac{1}{\phi}}}\right)\right]^{\frac{1}{\gamma}},$$

and then

$$\frac{\lambda x}{\gamma} = W_P\left(\frac{\lambda}{\gamma \alpha^{\frac{1}{\gamma}}}\left\{\log\frac{1}{1 - A^{\frac{1}{\phi}}}\right\}^{\frac{1}{\gamma}}\right).$$

Finally, we have

$$Q_X(p) = \frac{\gamma}{\lambda}\, W_{\mathrm{P}}\left(\frac{\lambda}{\gamma\alpha^{\frac{1}{\gamma}}}\left\{\log\frac{1}{1-A^{\frac{1}{\varphi}}}\right\}^{\frac{1}{\gamma}}\right), \qquad p \in (0,1), \tag{21}$$

where

$$A = A(p) = \frac{1-\eta-\sqrt{(1+\eta)^2-4p\eta}}{2\eta}\,, \quad \eta > 0\,.$$

Letting $H(\cdot)$ be any integrable function on the positive real line, we can write

$$\int_0^\infty H(x)\, f(x)dx = \int_0^1 H\left(Q_X(p)\right) dp. \tag{22}$$

In fact, for specific $H(\cdot)$ functions, the integral on the right-hand side of (22) can be more easily evaluated using the `ProductLog[z]` in *Mathematica* than that one on the left-hand side. □

6. Estimation

Several methods have been proposed in the literature for parameter estimation but the maximum likelihood approach is the most employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used to obtain confidence intervals for the model parameters. In this section, we consider the maximum likelihood estimation of the unknown parameters of the proposed distribution on the basis of complete samples. In order to estimate the parameters of the TGMW density function defined in equation (7), the log-likelihood function, given the observed sample x_i (for $i = 1, \ldots, n$), is maximized with respect to these parameters. It is given by

$$\ell(\theta) = n\{\log(\alpha) + \log(\varphi)\} + \sum_{i=1}^{n}\left\{\lambda x_i - \alpha x_i^{\gamma}\exp\left(\lambda x_i\right)\right\} + (\varphi-1)\sum_{i=1}^{n}\log\left[1-\exp\{-\alpha x_i^{\gamma}\exp\left(\lambda x_i\right)\}\right]$$
$$+ (\gamma-1)\sum_{i=1}^{n}\log\left(x_i\right) + \sum_{i=1}^{n}\log\left(\gamma+\lambda x_i\right) + \sum_{i=1}^{n}\log\left[1+\eta-2\left\{1-\exp(-\alpha\exp\{\lambda x_i\}x_i^{\gamma})\right\}^{\varphi}\eta\right].$$

We assume that the following standard regularity conditions for $\ell = \ell(\theta) = \ell(\alpha, \varphi, \gamma, \lambda, \eta)$ hold: i) The parameter space, say Θ, is open and ℓ has a global maximum in Θ; ii) For almost all x, the fourth-order log-likelihood derivatives with respect to the model parameters exist and are continuous in an open subset of Θ that contains the true parameter; iii) The support of X does not depend on unknown parameters; iv) The expected information matrix is positive definite and finite. These regularity conditions are not restrictive and hold for the models cited in this paper.

The nonlinear system of the log-likelihood equations resulting from equating the derivatives of $\ell(\theta)$ with respect to each parameter to zero is

$$\frac{\partial\ell(\theta)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\exp\left(\lambda x_i\right)x_i^{\gamma} - (1-\varphi)\sum_{i=1}^{n}\frac{\exp\{\lambda x_i - \exp\left(\lambda x_i\right)\alpha x_i^{\gamma}\}x_i^{\gamma}}{1-\exp\{-\exp\left(\lambda x_i\right)\alpha x_i^{\gamma}\}}$$
$$- \sum_{i=1}^{n}\frac{2\exp\left(\lambda x_i\right) - \alpha x_i^{\gamma}\exp(\lambda x_i)\left[1-\exp\{-\exp(\lambda x_i)\alpha x_i^{\gamma}\}\right]^{-1+\varphi}\eta\,\varphi\, x_i^{\gamma}}{1+\eta-2\,\eta\left[1-\exp\{-\exp(\lambda x_i)\alpha x_i^{\gamma}\}\right]^{\varphi}} = 0,$$

$$\frac{\partial\ell(\theta)}{\partial\varphi} = \frac{n}{\varphi} + \sum_{i=1}^{n}\log\left[1-\exp\{-\alpha\, x_i^{\gamma}\exp(\lambda x_i)\}\right]$$
$$- 2\,\eta\sum_{i=1}^{n}\frac{\left[1-\exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\right]^{\varphi}\log\left(1-\exp\{-\exp(\lambda x_i)\,\alpha\, x_i^{\gamma}\}\right]}{1+\eta-2\,\eta\left[1-\exp\{-\exp(\lambda x_i)\,\alpha\, x_i^{\gamma}\}\right]^{\varphi}} = 0,$$

$$\frac{\partial \ell(\theta)}{\partial \gamma} = \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{1}{\gamma+\lambda x_i} + \sum_{i=1}^{n} -\exp(\lambda x_i)\,\alpha\,\log(x_i)\,x_i^{\gamma}$$
$$+(-1+\varphi)\sum_{i=1}^{n}\frac{\exp\{(\lambda x_i)-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\,\alpha\,\log(x_i)\,x_i^{\gamma}}{1-\exp\{-\exp(\lambda x_i)\,\alpha\,x_i^{\gamma}\}}$$
$$-\sum_{i=1}^{n}\frac{2\exp\{\lambda x_i-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\left[1-\exp\{-\exp(\lambda x_i)\,\alpha\,x_i^{\gamma}\}\right]^{-1+\varphi}\alpha\,\eta\,\varphi\,\log(x_i)\,x_i^{\gamma}}{1+\eta-2\,\eta\left[1-\exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\right]^{\varphi}}=0,$$
$$\frac{\partial \ell(\theta)}{\partial \lambda} = (-1+\varphi)\sum_{i=1}^{n}\frac{\exp\{\lambda x_i-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\,\alpha x_i^{1+\gamma}}{1-\exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}} + \sum_{i=1}^{n}\frac{x_i}{\gamma+\lambda x_i} + \sum_{i=1}^{n}\left[x_i-\alpha x_i^{1+\gamma}\exp(\lambda x_i)\right]$$
$$-\sum_{i=1}^{n}\frac{2\exp\{\lambda x_i-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\left[1-\exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\right]^{-1+\varphi}\alpha\,\eta\,\varphi\,x_i^{1+\gamma}}{1+\eta-2\eta\left[1-\exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\right]^{\varphi}}=0,$$
$$\frac{\partial \ell(\theta)}{\partial \eta} = \sum_{i=1}^{n}\frac{1-2\left[1-\exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\right]^{\varphi}}{1+\eta-2\,\eta\left[1-\exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}\right]^{\varphi}}=0. \tag{23}$$

Although these equations cannot be solved analytically, a numerical solution can be determined by using computing packages. Iterative techniques such as Newton–Raphson type algorithms can be adopted to obtain $\hat{\theta}$. We employed the NLMixed procedure in SAS. The global maxima of the log-likelihood can be investigated by setting different starting values for the initial parameters. The information matrix is required for interval estimation. Expressions for the elements $J(\theta) = \{J_{rs}(\theta)\}$ for $r, s = \alpha, \gamma, \lambda, \varphi, \eta$ as the opposite of the 5×5 Hessian matrix of $\ell(\theta)$ are derived. As they are quite lengthy, the fifteen distinct elements of $J(\theta)$ are not included in the present paper. They can be obtained by differentiating $\ell(\theta)$ with respect to pairs of parameters and multiplying the results by -1, which can be readily achieved by making use of any symbolic computing packages. Alternatively, the results are available upon request from the second author. Under the aforementioned regularity conditions, the asymptotic distribution of $(\hat{\theta} - \theta)$ is $N_5(O, K(\theta)^{-1})$, where $K(\theta) = E\{J(\theta)\}$ is the expected information matrix. The approximate multivariate normal $N_5(O, J(\hat{\theta})^{-1})$ distribution, where $J(\hat{\theta})^{-1}$ is the inverse of the observed information matrix, can be used in practice to construct approximate confidence intervals for the parameters.

The following results pertain to the existence and uniqueness of the maximum likelihood estimates of φ and η.

Theorem 6.1. *Let us suppose that the parameters α, λ, γ and η are known. If $\eta \in (0, 1)$, then there exists a unique MLE of the parameter φ.*

Proof. To simplify the proof we will use the notation

$$y_i = 1 - \exp\{-\alpha x_i^{\gamma}\exp(\lambda x_i)\}.$$

Then, we obtain that

$$\frac{\partial^2 \ell(\theta)}{\partial \varphi^2} = -\frac{n}{\varphi^2} - 2\eta(1+\eta)\sum_{i=1}^{n}\frac{y_i^{\varphi}\{\log(y_i)\}^2}{\left(1+\eta-2\eta y_i^{\varphi}\right)^2}.$$

Since $\eta \in (0, 1)$, it follows that $\frac{\partial^2 \ell(\theta)}{\partial \varphi^2} < 0$, which means that $\frac{\partial \ell(\theta)}{\partial \varphi}$ is a decreasing function. Also we have that $\lim_{\varphi\to 0}\frac{\partial \ell(\theta)}{\partial \varphi} = \infty$ and $\lim_{\varphi\to\infty}\frac{\partial \ell(\theta)}{\partial \varphi} = \sum_{i=1}^{n}\log(y_i) < 0$, which proves the uniqueness. □

Theorem 6.2. *Let us suppose that the parameters α, λ, γ and η are known. If $\eta \in (-1, 0)$, then there exists at least one MLE of the parameter φ which lies in the interval* $\left[\frac{n}{-2\sum_{i=1}^{n}\log(y_i)}, \frac{n}{-\sum_{i=1}^{n}\log(y_i)}\right]$.

Proof. First, we have that

$$\frac{\partial \ell(\theta)}{\partial \varphi} = \frac{n}{\varphi} + \sum_{i=1}^{n} \log(y_i) - 2\,\eta \sum_{i=1}^{n} \frac{y_i^{\varphi} \log(y_i)}{1+\eta-2\,\eta y_i^{\varphi}};$$

since $0 < y_i < 1$ and $-1 < \eta < 0$, we obtain that $-2\,\eta \sum_{i=1}^{n} \frac{y_i^{\varphi} \log(y_i)}{1+\eta-2\,\eta y_i^{\varphi}} < 0$, which implies that

$$\frac{\partial \ell(\theta)}{\partial \varphi} < \frac{n}{\varphi} + \sum_{i=1}^{n} \log(y_i).$$

Thus, when $\varphi > \frac{n}{-\sum_{i=1}^{n} \log(y_i)}$, it follows that $\frac{\partial \ell(\theta)}{\partial \varphi}$ is negative.

Next, since $0 < \frac{-2\,\eta\, y_i^{\varphi}}{1+\eta-2\,\eta\, y_i^{\varphi}} < 1$, it follows that

$$\frac{-2\,\eta\, y_i^{\varphi} \log(y_i)}{1+\eta-2\,\eta\, y_i^{\varphi}} > \log(y_i),$$

which implies that

$$\frac{\partial \ell(\theta)}{\partial \varphi} > \frac{n}{\varphi} + 2\sum_{i=1}^{n} \log(y_i).$$

Finally, for $\varphi < \frac{n}{-2\sum_{i=1}^{n} \log(y_i)}$, we obtain that $\frac{\partial \ell(\theta)}{\partial \varphi}$ is positive. So, the proof follows from the continuity of the function $\frac{\partial \ell(\theta)}{\partial \varphi}$. □

Further, let us assume that the parameters $\alpha > 0$, $\gamma > 0$, $\lambda > 0$ and $\varphi > 0$ are known. The log-likelihood for η can be expressed as

$$\ell(\eta) = \sum_{i=1}^{n} \log(1+\eta-2\,\eta\, y_i^{\varphi}), \quad \eta \in [-1, 1],$$

where $y_1, \ldots, y_n$ can be considered known values in $(0, 1)$.

Under what conditions does the MLE $\hat{\eta}$ of η exist? The existence of the MLE $\hat{\eta}$ is intimately related to the y_i's, and so it is very difficult to establish some necessary and sufficient conditions. However, we can obtain sufficient conditions (depending on the observations) such as those cited in Theorem 6.3.

For example, $\hat{\eta}$ does not exist when $n = 1$. In this case, the function

$$g(\eta) = 1+\eta-2\,\eta\, y_1^{\varphi} = 1 + (1-2\,y_1^{\varphi})\eta$$

is a straight line with slope $1 - 2\,y_1^{\varphi}$, which is strictly increasing or decreasing if $y_1 < \left(\frac{1}{2}\right)^{1/\varphi}$ or $y_1 > \left(\frac{1}{2}\right)^{1/\varphi}$, respectively. Since $\log(x)$ is strictly increasing, we have that $\ell(\eta) = \log[g(\eta)]$ is strictly increasing or decreasing, which implies that $\ell(\eta)$ does not have a maximum in the open interval $(-1, 1)$.

For $n > 1$, the problem is more complex and $\hat{\eta}$ may or may not exist.

Theorem 6.3. *If* $\prod_{i=1}^{n}(1-y_i^{\varphi}) < \frac{1}{2^n}$ *and* $\prod_{i=1}^{n} y_i < \frac{1}{2^{n/\varphi}}$, *then the MLE* $\hat{\eta}$ *exists.*

Proof. We have $\lim_{\eta\to 1}\ell(\eta) = \sum_{i=1}^{n}\log[2(1-y_i^{\varphi})]$ and $\lim_{\eta\to -1}\ell(\eta) = \sum_{i=1}^{n}\log(2\,y_i^{\varphi})$. So, the conditions given in Theorem 6.3 imply that $\lim_{\eta\to 1}\ell(\eta) < 0$ and $\lim_{\eta\to -1}\ell(\eta) < 0$. Since $\ell(0) = 0$ and $\ell(\eta)$ is continuously differentiable and strictly concave $\left(\frac{\partial^2\ell(\eta)}{\partial\eta^2} < 0\right)$, there exists a unique maximum point in the interval $(-1, 1)$.

Henceforth, let $y_{(1)} = \min\{y_1, \ldots, y_n\}$ and $y_{(n)} = \max\{y_1, \ldots, y_n\}$. In what follows, we present results for the existence (or lack thereof) of the MLE $\hat{\eta}$ in terms of $y_{(1)}$ and $y_{(n)}$. □

The following result is a corollary of Theorem 6.3.

Corollary 6.4. *If* $y_{(1)} < \dfrac{1}{2^{n/\varphi}}$ *and* $y_{(n)} > \left(1 - \dfrac{1}{2^n}\right)^{1/\varphi}$*, then* $\hat{\eta}$ *exists.*

Since $y_i \in (0, 1)$, for $i = 1, \ldots, n$, we obtain

$$\prod_{i=1}^{n} y_i^{\varphi} < y_{(1)}^{\varphi}, \quad \prod_{i=1}^{n}(1 - y_i^{\varphi}) < 1 - y_{(n)}^{\varphi}.$$

Then, $y_{(1)} < \frac{1}{2^{n/\varphi}}$ and $y_{(n)} > \left(1 - \frac{1}{2^n}\right)^{1/\varphi}$ imply that the conditions given in Theorem 6.3 hold, which guarantee the existence of $\hat{\eta}$.

Theorem 6.5. *If* $y_{(n)} < \left(\dfrac{1}{2}\right)^{1/\varphi}$ *or* $y_{(1)} > \left(\dfrac{1}{2}\right)^{1/\varphi}$*, then* $\hat{\eta}$ *does not exist.*

Proof. In fact, in these instances, the straight lines

$$g_i(\eta) = 1 + (1 - 2\,y_i^{\varphi})\,\eta, \quad i = 1, \ldots, n$$

are strictly increasing if $y_{(n)} < \left(\frac{1}{2}\right)^{1/\varphi}$ or strictly decreasing if $y_{(1)} > \left(\frac{1}{2}\right)^{1/\varphi}$. Then, $\ell(\eta) = \sum_{i=1}^{n}\log[g_i(\eta)]$ will be strictly increasing or decreasing, which implies that $\ell(\eta)$ has no maximum point in $(-1, 1)$.

Finally, it is easy to check that U_η tends to $0.5\sum 1/y_i^{\varphi} - n$ when η goes to -1 and that U_η tends to $\sum\frac{(1-2y_i^{\varphi})}{2(1-y_i^{\varphi})}$ when η goes to 1. The first derivative of the score function is negative, and so the score function is decreasing. By performing some simulations of the y_i's, we could verify that, in certain instances, $U_\eta > 0$ when $\eta \to -1$ and $U_\eta < 0$ when $\eta \to 1$, which implies the uniqueness of $\hat{\eta}$. However, there are some situations, where both limits are negative or positive, and then there is no MLE of η.

In terms of approximation, the previous results indicate that conditions for the existence of the MLE $\hat{\eta}$ are $y_{(1)} \approx 0$ and $y_{(n)} \approx 1$. □

The Anderson-Darling (A^*) and Cramér-von Mises (W^*) statistics are adopted to compare the fitted models. These statistics are widely used to determine how closely a specific cdf fits the associated empirical distribution for a given data set. The smaller these statistics are, the better the fit is.

7. Applications

In this section, we compare the TGMW model with other related lifetime models, namely: the transmuted exponential–Weibull (TEW) [5], generalized modified Weibull (GMW) [4], Mcdonald Lomax (McLomax) [25], beta modified Weibull (BMW) [14] and transmuted exponentiated Weibull geometric (TEWG) [12] distributions. To do so, we make use of two real data sets: first, the carbon fibre data [7] and, secondly, the bladder cancer data [24]. More specifically, the fitted models are:

- The TEW density function [5]

$$f(x) = \left(\lambda + \beta\,k\,x^{k-1}\right)\exp\left\{-2\left(\lambda x + \beta\,x^k\right)\right\}\left\{2\alpha + (1-\alpha)\exp\left(\lambda x + \beta\,x^k\right)\right\},\ x > 0;$$

- The GMW density function [4]

$$f(x) = \varphi\,\alpha\,x^{\gamma-1}\,(\gamma+\lambda x)\,\exp\{\lambda x - \alpha x^{\gamma}\exp(\lambda x)\}\,[1-\exp\{-\alpha\,x^{\gamma}\exp(\lambda x)\}]^{\varphi-1}\,,\ x>0;$$

- The McLomax density function [25]

$$f(x) = \frac{c\,\alpha\,\beta^{\alpha}\,(\beta+x)^{-(\alpha+1)}}{\mathrm{Beta}\,(a\,c^{-1},b)}\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{a-1}\left[1-\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{c}\right]^{b-1},\ x>0;$$

- The BMW density function [14]

$$f(x) = \frac{\alpha\,x^{\gamma-1}\,(\gamma+\lambda x)\exp(\lambda x)\,[1-\exp\{-\alpha x^{\gamma}\exp(\lambda x)\}]^{a-1}\,\exp\{-b\,\alpha\,x^{\gamma}\exp(\lambda x)\}}{\mathrm{Beta}(a,b)},\ x>0;$$

- The TEWG density function [12]

$$f(x) = \theta\,\beta\,\alpha^{\theta}\,(1-p)\,x^{\theta-1}\,\exp\left\{-(\alpha x)^{\theta}\right\}\left[1-\exp\left\{-(\alpha x)^{\theta}\right\}\right]^{\beta-1}\left[1-p\left\{1-\exp\left\{-(\alpha x)^{\theta}\right\}\right\}^{\beta}\right]^{-2}$$
$$\times\left[(1+\lambda)-2\lambda\left\{\frac{(1-p)\left(1-\exp\left\{-(\alpha x)^{\theta}\right\}\right)^{\beta}}{1-p\,(1-\exp\{-(\alpha x)^{\theta}\})^{\beta}}\right\}\right],\ x>0.$$

7.1. *The Carbon fibre data*

The first data set which is uncensored pertains to the breaking stress of carbon fibres (in Gba) as reported in Cordeiro *et al.* [7].

7.2. *The bladder cancer data*

The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in Lee and Wang [24].

The estimated pdf's and cdf's of the TGMW model are plotted in Figures 4 and 5 for the carbon fibres and cancer data, respectively. The estimates of the parameters as well as the values of the Anderson-Darling (A^{*}) and Cramér-von Mises (W^{*}) statistics are listed in Tables 2 to 5. We note that the TGMW model provides the best fit for both data sets.

Table 2: MLEs (standard errors in parentheses) for the carbon fibres

Distributions	Parameter	estimates			
TEW (λ, β, k, α)	0.01297	0.00581	4.11180	0.67244	
	(0.01376)	(0.00397)	(0.50670)	(0.37129)	
GMW (φ, α, γ, λ)	5.49894	0.43639	0.14811	0.51628	
	(8.02208)	(0.64986)	(0.53839)	(0.16932)	
McLomax (α, β, a, b, c)	4.01844	44.9998	3.37645	1499.98	5.43418
	(16.155)	(177.75)	(0.79071)	(7941.2)	(3.49922)
BMW (α, γ, λ, a, b)	0.44730	0.13899	0.49618	5.87258	1.12967
	(0.72868)	(0.54698)	(0.46150)	(12.2267)	(2.95269)
TEWG (α, θ, β, p, λ)	59.2556	0.45587	1.42577	0.99991	-0.44753
	(27.5648)	(0.03366)	(1.60102)	(0.00937)	(0.49717)
TGMW (α, φ, γ, λ, η)	0.19212	3.31948	0.27486	0.58561	0.67440
	(0.40783)	(4.54722)	(0.82386)	(0.23501)	(0.37119)

Table 3: Goodness-of-fit statistics for the carbon fibres

Distributions	A^*	W^*
TEW (λ, β, k, α)	0.33372	0.05325
GMW(φ, α, γ, λ)	0.38543	0.06279
McLomax (α, β, a, b, c)	0.49648	0.08398
BMW (α, γ, λ, a, b)	0.38423	0.06261
TEWG (α, θ, β, p, λ)	0.77199	0.12016
TGMW (α, φ, γ, λ, η)	**0.33211**	**0.05279**

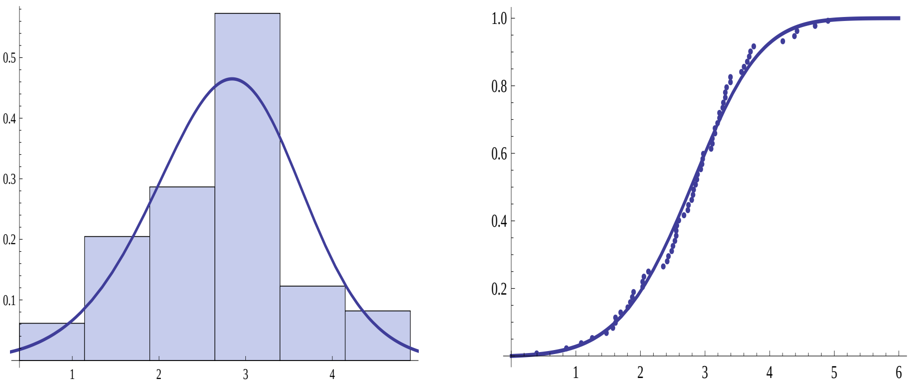

Figure 5: Left panel: The fitted TGMW density superimposed on the histogram for the carbon fibres data. Right panel: The estimated TGMW cdf and empirical cdf.

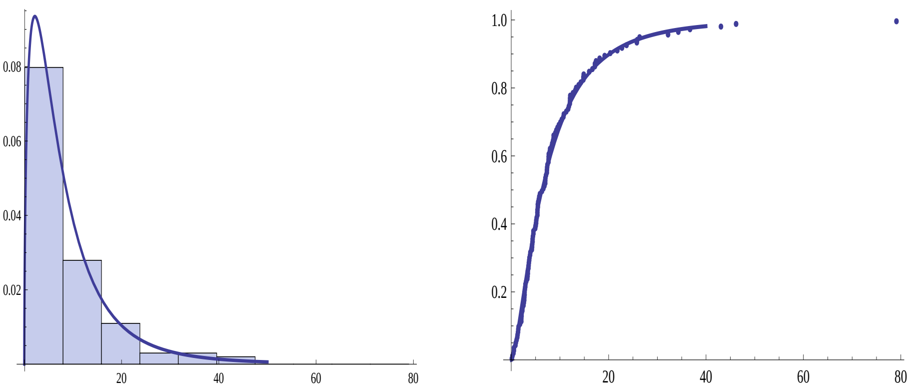

Figure 6: Left panel: The estimated TGMW density superimposed on the histogram for the bladder cancer data. Right panel: The estimated TGMW cdf and empirical cdf.

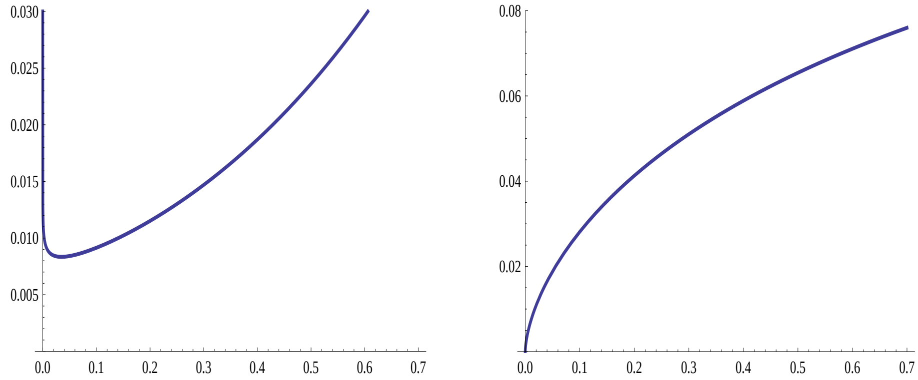

Figure 7: Left panel: The estimated TGMW hrf for the carbon fibres data set. Right panel: The estimated TGMW hrf for the bladder cancer data set.

Table 4: MLEs (standard errors in parentheses) for the bladder cancer data

Distributions	Parameter	estimates			
TEW (λ, β, k, α)	1.08×10^{-10}	0.04783	1.13331	0.74492	
	(0.07844)	(0.07216)	(0.14413)	(0.20247)	
GMW(φ, α, γ, λ)	2.79601	0.45369	0.65441	5.8×10^{-13}	
	(1.85772)	(0.37182)	(0.24811)	(0.00628)	
McLomax (α, β, a, b, c)	0.8085	11.2929	1.5060	4.1886	2.1046
	(3.364)	(15.818)	(0.243)	(25.029)	(3.079)
BMW (α, γ, λ, a, b)	0.46965	0.66613	5.8×10^{-13}	2.73477	0.90825
	(0.47875)	(0.31225)	(0.00639)	(2.02018)	(1.52196)
TEWG (α, θ, β, p, λ)	1119.9	0.20963	23.2028	0.91371	-0.87193
	(574.98)	(0.01021)	(37.768)	(0.14151)	(0.21604)
TGMW (α, φ, γ, λ, η)	0.25215	2.24129	0.72431	3.4×10^{-11}	0.72252
	(0.31749)	(1.74023)	(0.38549)	(0.00795)	(0.35566)

Table 5: Goodness-of-fit statistics for the bladder cancer data

Distributions	A_0^*	W_0^*
TEW (λ, β, k, α)	0.56339	0.08825
GMW(φ, α, γ, λ)	0.27198	0.04050
McLomax (α, β, a, b, c)	1.81435	0.3550
BMW (α, γ, λ, a, b)	0.27197	0.04051
TEWG (α, θ, β, p, λ)	0.40445	0.05427
TGMW (α, φ, γ, λ, η)	**0.18733**	**0.02732**

8. Conclusions

There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models in order to improve the modeling of survival data. As a result, significant progress has been made towards the generalization of the traditional Weibull model. In this paper, we propose a five–parameter model named the *transmuted generalized modified Weibull* (TGMW) distribution, which is obtained by applying the transmuted generalized technique to the exponentiated modified Weibull modified model. The new model extends several important lifetime distributions. We studied some of its statistical properties and obtained representations of the positive, negative and factorial moments, as well as the reliability, quantile function and the density of the order statistics. The proposed distribution as applied to two actual data sets turned out to provide better fits than other competing lifetime models. The computing code is available from the second author upon request. The distributional results developed in this paper should find numerous applications in various fields of scientific investigation such as reliability theory, hydrology, biostatistics, meteorology, engineering and survival analysis.

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References

[1] Aryal GR, Tsokos, CP. Transmuted Weibull distribution: a generalization of the Weibull probability distribution. Eur. J. Pure Appl. Math. 4 (2011) 89–102.
[2] Lai CD, Xie M, Murthy DNP. A modified Weibull distribution. IEEE Transactions on Reliability. 52 (2003) 33–37.
[3] Ghitany ME, Al-Hussaini EK, Al-Jarallah RA. Marshall–Olkin extended Weibull distribution and its application to censored data. J. Appl. Statist. 32 (2005) 1025–1034.
[4] Carrasco JMF, Edwin MM Ortega, Cordeiro GM. A generalized modified Weibull distribution for lifetime modeling. Comput. Stat. Data Anal. 53 (2008) 450–462.
[5] Saboor A, Kamal M, Ahmad M. The transmuted exponential-Weibull distribution with applications. Pak. J. Statist. 31 (2015) 229–250.
[6] Cordeiro GM, Edwin MM Ortega, Nadarajah S. The Kumaraswamy Weibull distribution with application to failure data. J. Franklin Inst. 347 (2010) 1399–1429.
[7] Cordeiro GM, Edwin MM Ortega, Lemonte AJ. The exponential–Weibull lifetime distribution. J. Statist. Comput. Simulation. 84 (2014) 2592–2606.
[8] Peng X, Yan Z. Estimation and application for a new extended Weibull distribution. Reliab. Eng. Syst. Safe. 121 (2014) 34–42.
[9] Khan MN. The modified beta Weibull distribution. Hacettepe Journal of Mathematics and Statistics. 44 (2015) 1553–1568.
[10] Tojeiro C, Louzada F, Roman M, Borges P. The complementary Weibull geometric distribution. J. Statist. Comput. Simulation. 84 (2014) 1345-1362.
[11] Almalki SJ, Yuan J. A new modified Weibull distribution. Reliab. Eng. Syst. Safe. 111 (2013) 164–170.
[12] Saboor A, Elbatal I, Cordeiro GM. The transmuted exponentiated Weibull geometric distribution: theory and applications. Hacettepe Journal of Mathematics and Statistics. 45 (2016) 973–987.
[13] Jiang R, Murthy DNP. Mixture of Weibull distributions–parametric characterization of failure rate function. Appl. Stoch. Model. D. A. 14 (1998) 47–65.
[14] Silva GO, Edwin MM Ortega and Cordeiro GM. The beta modified Weibull distribution. Lifetime. Data. Anal. 16 (2010) 409–430.
[15] Shaw W, Buckley I. The Alchemy of Probability Distributions: Beyond Gram-Charlier Expansions, and a Skew -Kurtotic–Normal Distribution from a Rank Transmutation Map. 2007; Research report.
[16] Mudholkar GS, Srivastava DK, Friemer M. The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. Technometrics. 37 (1995) 436–445.
[17] Mudholkar GS, Srivastava DK, Kollia GD. A generalization of the Weibull distribution with application to the analysis of survival data. Journal of the American Statistical Association. 91 (1996) 1575–1583.
[18] Gupta RD, Kundu D. Generalized exponential distributions. Australian and New Zealand Journal of Statistics. 41 (1999) 173–188.
[19] Gupta RD, Kundu D. Exponentiated exponential distribution: an alternative to gamma and Weibull distributions. Biometrical Journal. 43 (2001) 117–130.
[20] Kundu D, Rakab MZ. Generalized Rayleigh distribution: different methods of estimation. Computational Statistics and Data Analysis. 49 (2005) 187–200.
[21] Lai CD, Moore T, Xie M. The beta integrated model. Proceedings International Workshop on Reliability Modeling and Analysis - From Theory to Practice. (1998) 153–159.
[22] http://dlmf.nist.gov/4.13
[23] Gradshteyn IS, Ryzhik IM. Table of Integrals, Series and Products (sixth edition). 2000; San Diego: Academic Press.
[24] Lee E, Wang J. Statistical Methods for Survival Data Analysis. Wiley & Sons: New York; 2003.
[25] Lemonte AJ, Cordeiro GM. An extended Lomax distribution. Statistics: A Journal of Theoretical and Applied Statistics. 47 (2013) 800–816.